

# Engineering Sciences

## Section A. Engineering Mathematics

- Linear Algebra
- Calculus
- Complex Analysis
- Vector Calculus
- Differential Equations
- Probability - I
- Probability (continued)
- Numerical Methods  
(above topics are successfully completed in Mathematics part)

## Section B. Computational Science

- Truncation errors,
- round off errors and Their propagation
- Elementary Vector Analysis
- Lines, Planes, and Vectors
- Multiple Integration
- Multi-Variable Chain Rule

## Truncation Errors

Applying a finite difference operator has the side effect of introducing truncation errors into the solution. In this section we discuss how truncation enters the system and how it affects it.

- Truncation and Dispersion<sup>3.5</sup>
- Consequences and Interpretation

## Truncation and Dispersion

Earlier we presented a generic equation describing the conservation laws. This allows us to extract information of the numeric properties of the system quite easily, as the analysis of the one generic equation can be propagated to all its special cases. We will not make any assumptions on the drains and sources in the system, and hence do not include any such in this presentation, instead we consider the simple conserved system:

$$\frac{\partial \Gamma}{\partial t} = -\nabla \cdot \Gamma \mathbf{u}.$$

Now, to make the analysis easier we initially consider the one dimensional case with constant velocity  $u$ ,

$$\frac{\partial \Gamma}{\partial t} + u \frac{\partial \Gamma}{\partial x} = 0.$$

In a previous section we discussed the type and order of the difference operators, in the following we will limit the analysis to a simple centered difference operator, but as we will see the analysis is straight-forward, and may be performed using operators of any order. In continuous time,

discrete space, with  $\Gamma_i$  designating  $\Gamma$  evaluated at grid node  $i$ , we find

$$\frac{\partial \Gamma}{\partial t} + u \frac{\Gamma_{i+1} - \Gamma_{i-1}}{2\Delta x} = 0.$$

As we have noted earlier, this discretisation implies losing some higher-order terms. To see why this happens we will express the above approximation by the Taylor expansion of the terms

involved, as we approximate  $\Gamma_{i+1}$  and  $\Gamma_{i-1}$  by respectively expanding forward and backward from the value  $\Gamma_i$  by the distance  $\Delta x$  between any two grid nodes:

$$\begin{aligned}
\Gamma_{i+1} &= \Gamma_i + \frac{\Delta x}{1!} \frac{\partial \Gamma_i}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Gamma_i}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 \Gamma_i}{\partial x^3} + O(\Delta x^4) \\
\Gamma_{i-1} &= \Gamma_i + \frac{-\Delta x}{1!} \frac{\partial \Gamma_i}{\partial x} + \frac{(-\Delta x)^2}{2!} \frac{\partial^2 \Gamma_i}{\partial x^2} + \frac{(-\Delta x)^3}{3!} \frac{\partial^3 \Gamma_i}{\partial x^3} + O(\Delta x^4) \\
&= \Gamma_i - \frac{\Delta x}{1!} \frac{\partial \Gamma_i}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 \Gamma_i}{\partial x^2} - \frac{\Delta x^3}{3!} \frac{\partial^3 \Gamma_i}{\partial x^3} + O(\Delta x^4) \\
&0 \\
\Gamma_{i+1} - \Gamma_{i-1} &= 2 \frac{\Delta x}{1!} \frac{\partial \Gamma_i}{\partial x} + 2 \frac{\Delta x^3}{3!} \frac{\partial^3 \Gamma_i}{\partial x^3} + O(\Delta x^5)
\end{aligned}$$

By inserting into the continuous time, discrete space approximation we find the equation we are *actually* solving

$$\frac{\partial \Gamma}{\partial t} + u \left( \frac{\partial \Gamma}{\partial x} + \frac{\Delta x^2}{3!} \frac{\partial^3 \Gamma}{\partial x^3} + O(\Delta x^4) \right) = 0.$$

### Consequences and Interpretation

To give an impression of what this means for the solution, we now let  $\Gamma$  represent a simple sinusoidal wave in time and space. We then isolate the temporal frequency and calculate the wave speed, from which we can gain some insight into how the system evolves. The wave can be described as  $\Gamma = e^{ikx - i\omega t}$  which enables us to extend the argument, at least to periodic solutions as we simply approximate the shape through Fourier series. By simple insertion we find

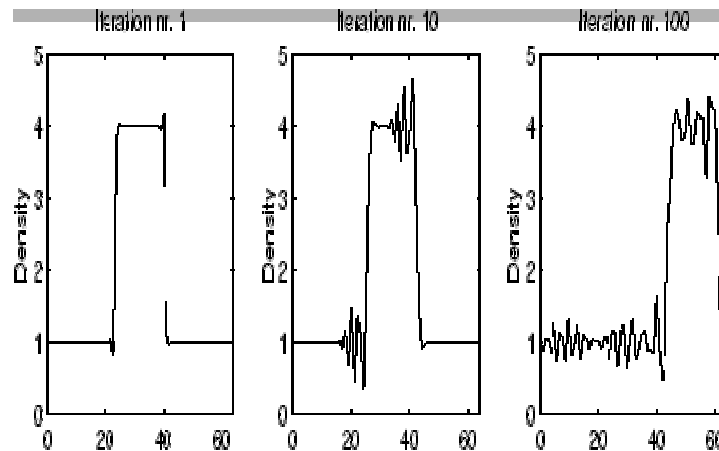
$$\begin{aligned}
&0 \\
&\frac{\partial \Gamma}{\partial t} + u \left( \frac{\partial \Gamma}{\partial x} + \frac{\Delta x^2}{3!} \frac{\partial^3 \Gamma}{\partial x^3} + \dots + \frac{\Delta x^{(n-1)}}{n!} \frac{\partial^n \Gamma}{\partial x^n} + \dots \right) = 0 \\
&0 \\
&-i\omega e^{ikx - i\omega t} + u \left( ik e^{ikx - i\omega t} - i \frac{\Delta x^2}{3!} k^3 e^{ikx - i\omega t} + \dots \right. \\
&\quad \left. + (-1)^n i \frac{\Delta x^{(n-1)}}{n!} k^n e^{ikx - i\omega t} + \dots \right) = 0 \\
&0 \\
&e^{ikx - i\omega t} i \left( -\omega + u \left( k - \frac{\Delta x^2}{3!} k^3 + \dots + (-1)^n \frac{\Delta x^{(n-1)}}{n!} k^n + \dots \right) \right) = 0 \\
&0 \\
&\omega = u k \left( 1 - \frac{\Delta x^2}{3!} k^2 + \dots + (-1)^n \frac{\Delta x^n}{n!} k^n + \dots \right).
\end{aligned}$$

With spatial wave length  $\lambda = \frac{2\pi}{k}$  and time frequency  $T = \frac{2\pi}{\omega}$  we find the wave speed

$$s = \frac{\lambda}{T} = \frac{\omega}{k} = u \left( 1 - \frac{(k\Delta x)^2}{3!} \dots + (-1)^n \frac{(k\Delta x)^n}{n!} + \dots \right),$$

which converges as the faculty operator is stronger than the power. We see that the speed at which a solution propagates is dependent on the spatial frequency, and that the most significant term enters with a negative coefficient, causing a slow down of frequencies. At constant grid spacing  $\Delta x$ , and as a low value of  $k$  models a high spatial frequency wave, we can now see high frequency modes will move slower than low frequency ones. This means that over time a solution consisting of a broad range of frequencies will change shape, as high mode frequencies will be left behind. This is especially evident in the solution of the square wave, which has non-vanishing amplitude in all frequency modes, and hence all terms in the Taylor expansion are of importance. This is the origin of the dispersion phenomena mentioned in an earlier section. In our case this is especially worrying, as we have already seen, shocks may evolve naturally in the system, and hence the near discontinuities in the shock fronts result in high order frequencies dominating the system. This can be seen in figure 3.5

**Figure 3.5:** Constant velocity propagation of a steep wave using a naive implementation of the difference operator.



, where we see how such a system evolves. We see that the truncation results in dispersion as high frequency modes are left behind by the bulk of the wave causing numerical instability due to the steep gradients flooding the solution. At roughly iteration number 130 the valley immediately following the hill became negative, crashing the simulation. As a rule of thumb truncation is of no immediate concern if the shock is resolved on the grid, i.e. the wave length of the highest mode frequency in the shock is a number of times longer than the grid spacing  $\Delta x$ , as is also hinted by the term  $\frac{\Delta x^n}{n!} \frac{\partial^n \Gamma_i}{\partial x^n}$  entering under the power operator.

Another view at the problem is noticing that in the difference operation of order  $n$  we truncated

$$\frac{\Delta x^n}{n!} \frac{\partial^n \Gamma_i}{\partial x^n}$$

terms from the Taylor series from the order  $n$  and up, terms for which the coefficients  $\frac{\Delta x^n}{n!}$

was in some sense small, under the assumption that the differential operator  $\frac{\partial^n \Gamma_i}{\partial x^n}$  is in some sense limited, making all the truncated terms insignificant for the solution. But at near

discontinuities this is *not* the case, as  $\frac{\partial^n \Gamma_i}{\partial x^n}$  becomes of significance, breaking the assumption and infecting the solution.

The net result is that the truncation error will be of relevance in shock regions, and that we cannot approximate the solution correctly in an extreme shock-region, no matter the order of the difference operator. If ignored this will in many cases cause the discrete approximations to explode, and in return drown the correct solution in numerical noise.

But what if we could avoid the discontinuities altogether?

## Round off Error

The error produced in a computation by rounding results at one or more intermediate steps, resulting in a result different from that which would be obtained using exact numbers. The most common problems resulting from roundoff error occur either when many steps are involved with rounding occurring at each step, when two quantities very close to each other are subtracted, or when a number is divided by a number which is close to zero.

An egregious example of roundoff error is provided by a short-lived index devised at the Vancouver stock exchange (McCullough and Vinod 1999). At its inception in 1982, the index was given a value of 1000.000. After 22 months of recomputing the index and truncating

to three decimal places at each change in market value, the index stood at 524.881, despite the fact that its "true" value should have been 1009.811.

Other sorts of roundoff error can also occur. A notorious example is the fate of the Ariane rocket launched on June 4, 1996 (European Space Agency 1996). In the 37th second of flight, the inertial reference system attempted to convert a 64-bit floating-point number to a 16-bit number, but instead triggered an overflow error which was interpreted by the guidance system as flight data, causing the rocket to veer off course and be destroyed.

The Patriot missile defense system used during the Gulf War was also rendered ineffective due to roundoff error (Skeel 1992, U.S. GAO 1992). The system used an integer timing register which was incremented at intervals of 0.1 s. However, the integers were converted to decimal numbers by multiplying by the binary approximation of 0.1,

$$0.00011001100110011001100_2 = \frac{209715}{2097152}.$$

As a result, after 100 hours ( $3.6 \times 10^6$  ticks), an error of

$$\left(\frac{1}{10} - \frac{209715}{2097152}\right) (3600 \cdot 100 \cdot 10) = \frac{5625}{16384} \approx 0.3433 \text{ second}$$

had accumulated. This discrepancy caused the Patriot system to continuously recycle itself instead of targeting properly. As a result, an Iraqi Scud missile could not be targeted and was allowed to detonate on a barracks, killing 28 people.

## Error Propagation

Given a formula  $y = f(x)$  with an absolute error in  $x$  of  $dx$ , the absolute error is  $dy$ . The relative error is  $dy/y$ . If  $x = f(u, v, \dots)$ , then

$$x_i - \bar{x} = (u_i - \bar{u}) \frac{\partial x}{\partial u} + (v_i - \bar{v}) \frac{\partial x}{\partial v} + \dots, \quad (1)$$

where  $\bar{x}$  denotes the mean, so the sample variance is given by

$$s_x^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2 \quad (2)$$

$$= \frac{1}{N-1} \sum_{i=1}^N \left[ (u_i - \bar{u})^2 \left(\frac{\partial x}{\partial u}\right)^2 + (v_i - \bar{v})^2 \left(\frac{\partial x}{\partial v}\right)^2 + 2(u_i - \bar{u})(v_i - \bar{v}) \left(\frac{\partial x}{\partial u}\right) \left(\frac{\partial x}{\partial v}\right) + \dots \right] \quad (3)$$

The definitions of variance and covariance then give

$$s_u^2 \equiv \frac{1}{N-1} \sum_{i=1}^N (u_i - \bar{u})^2 \quad (4)$$

$$s_v^2 \equiv \frac{1}{N-1} \sum_{i=1}^N (v_i - \bar{v})^2 \quad (5)$$

$$s_{uv} \equiv \frac{1}{N-1} \sum_{i=1}^N (u_i - \bar{u})(v_i - \bar{v}) \quad (6)$$

(where  $s_{ii} \equiv s_i^2$ ), so

$$s_x^2 = s_u^2 \left( \frac{\partial x}{\partial u} \right)^2 + s_v^2 \left( \frac{\partial x}{\partial v} \right)^2 + 2 s_{uv} \left( \frac{\partial x}{\partial u} \right) \left( \frac{\partial x}{\partial v} \right) + \dots \quad (7)$$

If  $u$  and  $v$  are uncorrelated, then  $s_{uv} = 0$  so

$$s_x^2 = s_u^2 \left( \frac{\partial x}{\partial u} \right)^2 + s_v^2 \left( \frac{\partial x}{\partial v} \right)^2. \quad (8)$$

Now consider addition of quantities with errors. For  $x = au \pm bv$ ,  $\partial x / \partial u = a$  and  $\partial x / \partial v = \pm b$ , so

$$s_x^2 = a^2 s_u^2 + b^2 s_v^2 \pm 2ab s_{uv}. \quad (9)$$

For division of quantities with  $x = \pm au/v$ ,  $\partial x / \partial u = \pm a/v$  and  $\partial x / \partial v = \mp au/v^2$ , so

$$s_x^2 = \frac{a^2}{v^2} s_u^2 + \frac{a^2 u^2}{v^4} s_v^2 - 2 \frac{a}{v} \frac{au}{v^2} s_{uv}. \quad (10)$$

Dividing through by  $x^2$  and rearranging then gives

$$\left( \frac{s_x}{x} \right)^2 = \frac{a^2}{v^2} \frac{v^2}{a^2 u^2} s_u^2 + \frac{a^2 u^2}{v^4} \frac{v^2}{a^2 u^2} s_v^2 - 2 \left( \frac{a}{v} \right) \left( \frac{au}{v^2} \right) s_{uv} \quad (11)$$

$$(12)$$

$$= \left( \frac{s_u}{u} \right)^2 + \left( \frac{s_v}{v} \right)^2 - 2 \left( \frac{s_{uv}}{u} \right) \left( \frac{s_{uv}}{v} \right). \quad (13)$$

For exponentiation of quantities with

$$x = a^{\pm b u} = (e^{\ln a})^{\pm b u} = e^{\pm b (\ln a) u}, \quad (14)$$

and

$$\frac{\partial x}{\partial u} = \pm b (\ln a) e^{\pm b \ln a u} = \pm b (\ln a) x, \quad (15)$$

so

$$s_x = s_u b (\ln a) x \quad (16)$$

$$\frac{s_x}{x} = b \ln a s_u. \quad (17)$$

If  $a = e$ , then

$$\frac{s_x}{x} = b s_u. \quad (18)$$

For logarithms of quantities with  $x = a \ln (\pm b u)$ ,  $\partial x / \partial u = a (\pm b) / (\pm b u) = a / u$ , so

$$s_x^2 = s_u^2 \left( \frac{a^2}{u^2} \right) \quad (19)$$

$$s_x = a \frac{s_u}{u}. \quad (20)$$

For multiplication with  $x = \pm a u v$ ,  $\partial x / \partial u = \pm a v$  and  $\partial x / \partial v = \pm a u$ , so

$$s_x^2 = a^2 v^2 s_u^2 + a^2 u^2 s_v^2 + 2 a^2 u v s_{uv} \quad (21)$$

$$\left( \frac{s_x}{x} \right)^2 = \frac{a^2 v^2}{a^2 u^2 v^2} s_u^2 + \frac{a^2 u^2}{a^2 u^2 v^2} s_v^2 + \frac{2 a^2 u v}{a^2 u^2 v^2} s_{uv} \quad (22)$$

$$= \left( \frac{s_u}{u} \right)^2 + \left( \frac{s_v}{v} \right)^2 + 2 \left( \frac{s_{uv}}{u} \right) \left( \frac{s_{uv}}{v} \right). \quad (23)$$

For powers, with  $x = a u^{\pm b}$ ,  $\partial x / \partial u = \pm a b u^{\pm b - 1} = \pm b x / u$ , so

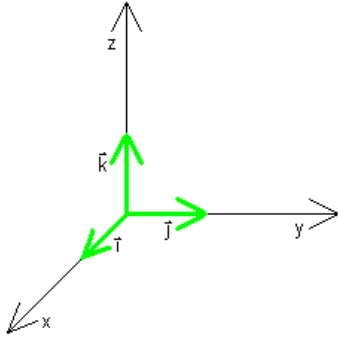
$$s_x^2 = s_u^2 \frac{b^2 x^2}{u^2} \quad (24)$$

$$\frac{s_x}{x} = b \frac{s_u}{u}.$$

## Elementary Vector Analysis

In order to measure many physical quantities, such as force or velocity, we need to determine both a magnitude and a direction. Such quantities are conveniently represented as vectors.





The direction of a vector  $\mathbf{v}$  in 3-space is specified by its components in the x, y, and z directions, respectively:

$(x,y,z)$  or  $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ,

$\mathbf{i} = (1,0,0)$ $\mathbf{j} = (0,1,0)$ $\mathbf{k} = (0,0,1)$
--

where  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are the **coordinate vectors** along the x-, y-, and z-axes.

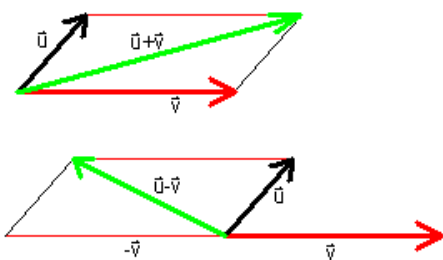
The magnitude of a vector  $\mathbf{v} = (x,y,z)$ , also called its length or **norm**, is given by

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

## Notes

- Vectors can be defined in any number of dimensions, though we focus here only on 3-space.
- drawing a vector in 3-space, where you position the vector is unimportant; the vector's essential properties are just its magnitude and its direction. Two vectors are **equal** if and only if corresponding components are equal.
- A vector of norm 1 is called a **unit vector**. The coordinate vectors are examples of unit vectors.
- The zero vector,  $\mathbf{0} = (0,0,0)$ , is the only vector with magnitude 0.

## Basic Operations on Vectors

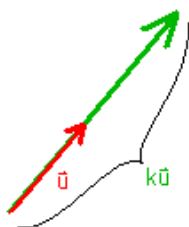


To add or subtract vectors  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ ,

add or subtract the corresponding coordinates:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$



To multiply vector  $\mathbf{u}$  by a scalar  $k$ , multiply each coordinate of  $\mathbf{u}$  by  $k$ :

$$k \mathbf{u} = (ku_1, ku_2, ku_3)$$

### Example

The vector  $\mathbf{v} = (2, 1, -2) = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  has magnitude

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3.$$

Thus, the vector  $\frac{1}{3}\mathbf{v} = (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$  is a unit vector in the same direction as  $\mathbf{v}$ .

In general, for  $\mathbf{v} \neq \mathbf{0}$ , we can scale (or **normalize**)  $\mathbf{v}$  to the unit vector  $\mathbf{v}/\|\mathbf{v}\|$  pointing in the same direction as  $\mathbf{v}$ .

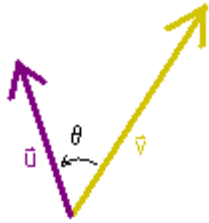
### Dot Product

Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ . The **dot product**  $\mathbf{u} \cdot \mathbf{v}$  (also called the **scalar product** or **Euclidean inner product**) of  $\mathbf{u}$  and  $\mathbf{v}$  is defined in two distinct (though equivalent) ways:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \quad \mathbf{u} \cdot \mathbf{v} \text{ is a number}$$

or

$$\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0} \quad \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \quad \text{where } 0 \leq \theta \leq \pi \text{ is the angle}$$



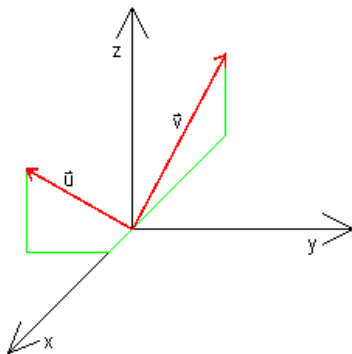
between  $\mathbf{u}$  and  $\mathbf{v}$

### Properties of the Dot Product

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$
- $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$

See if you can verify each of these!

### Example



If  $\mathbf{u} = (1, -2, 2)$  and  $\mathbf{v} = (-4, 0, 2)$ , then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (1)(-4) + (-2)(0) + (2)(2) \\
&= -4 + 0 + 4 \\
&= 0
\end{aligned}$$

Using the second definition of the dot product with  $\|\mathbf{u}\| = 3$  and  $\|\mathbf{v}\| = 2\sqrt{5}$ ,

$$\mathbf{u} \cdot \mathbf{v} = 0 = 6\sqrt{5} \cos \theta$$

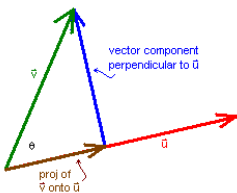
so  $\cos \theta = 0$ , yielding  $\theta = \pi/2$ .

Though we might not have guessed it,  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular to each other!

In general,

Two non-zero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular (or **orthogonal**) if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

### Projection of a Vector



It is often useful to resolve a vector  $\mathbf{v}$  into the sum of vector components parallel and perpendicular to a vector  $\mathbf{u}$ .

Consider first the parallel component, which is called the **projection of  $\mathbf{v}$  onto  $\mathbf{u}$** . This projection should be in the direction of  $\mathbf{u}$  and should have magnitude  $\|\mathbf{v}\| \cos \theta$ , where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . Let's normalize  $\mathbf{u}$  to  $\mathbf{u}/\|\mathbf{u}\|$  and then scale this by the magnitude  $\|\mathbf{v}\| \cos \theta$ :

$$\text{Projection of } \mathbf{v} \text{ onto } \mathbf{u} = (\|\mathbf{v}\| \cos \theta) \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

$$\begin{aligned} & \|\mathbf{v}\| \|\mathbf{u}\| \cos\theta \\ &= \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \\ &= \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \end{aligned}$$

The perpendicular vector component of  $\mathbf{v}$  is then just the difference between  $\mathbf{v}$  and the projection of  $\mathbf{v}$  onto  $\mathbf{u}$ .

In summary,

$\text{projection of } \mathbf{v} \text{ onto } \mathbf{u}: = \frac{\mathbf{v} \cdot \mathbf{u}}{\ \mathbf{u}\ ^2} \mathbf{u}$
$\text{vector component of } \mathbf{v} \text{ perpendicular to } \mathbf{u}: = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{\ \mathbf{u}\ ^2} \mathbf{u}$

## Cross Product

Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ . The **cross product**  $\mathbf{u} \times \mathbf{v}$  yields a vector perpendicular to both  $\mathbf{u}$  and  $\mathbf{v}$  with direction determined by the right-hand rule. Specifically,

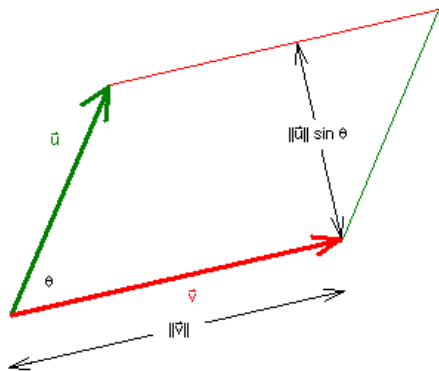
$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$\mathbf{u} \times \mathbf{v}$  is a vector

It can also be shown that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta \text{ for } \mathbf{u} \neq \mathbf{0}, \text{ for } \mathbf{v} \neq \mathbf{0}$$

where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .



Thus, the magnitude  $\|\mathbf{u} \times \mathbf{v}\|$  gives the area of the parallelogram formed by  $\mathbf{u}$  and  $\mathbf{v}$ .

As implied by the geometric interpretation,

Non zero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are parallel if and only if  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ .

### Properties of the Cross Product

- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Again, see if you can verify each of these.

In the following Exploration, select values for the components of  $\mathbf{u}$  and  $\mathbf{v}$ . You will see  $\mathbf{u} \cdot \mathbf{v}$  and  $\mathbf{u} \times \mathbf{v}$  computed and  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{u} \times \mathbf{v}$  displayed on a coordinate system.

### Exploration

#### Key Concepts

Let  $\mathbf{u} = (u_1, u_2, u_3)$  and  $\mathbf{v} = (v_1, v_2, v_3)$ .

- **Basic Operations, Norm of a vector**

$$\mathbf{u} + \mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$$

$$\mathbf{u} - \mathbf{v} = (u_1-v_1, u_2-v_2, u_3-v_3)$$

$$k \mathbf{u} = (ku_1, ku_2, ku_3)$$

$$\|\mathbf{v}\| = \sqrt{x^2+y^2+z^2}$$

- **Dot Product**

$$\mathbf{u} \cdot \mathbf{v}$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

if

$$\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta = \mathbf{u} \cdot \mathbf{v} \quad \text{if } \mathbf{u} \perp \mathbf{v}, \mathbf{u} \cdot \mathbf{v} = 0 \text{ or } \mathbf{v} = \mathbf{0}$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$

For  $\mathbf{u} \perp \mathbf{v}$ ,  $\mathbf{u} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{u}$  is orthogonal to  $\mathbf{v}$ .

- **Projection of a Vector**

$$\mathbf{v} \cdot \mathbf{u}$$

$$\text{projection of } \mathbf{v} \text{ onto } \mathbf{u}: = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u}$$

$$\|\mathbf{u}\|^2$$

$$\mathbf{v} \cdot \mathbf{u}$$

$$\text{vector component of } \mathbf{v} \text{ perpendicular to } \mathbf{u}: = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u}$$

$$\|\mathbf{u}\|^2$$

- **Cross Product**

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \text{ for } \mathbf{u} \neq \mathbf{0}, \text{ for } \mathbf{v} \neq \mathbf{0}$$

where  $0 < \theta < \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

For  $\mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0}, \mathbf{u} \times \mathbf{v} = \mathbf{0}$  if and only if  $\mathbf{u}$  is parallel to  $\mathbf{v}$ .

## Lines, Planes, and Vectors

In this tutorial, we will use vector methods to represent lines and planes in 3-space.

### Displacement Vector

The displacement vector  $\mathbf{v}$  with initial point  $(x_1, y_1, z_1)$  and terminal point  $(x_2, y_2, z_2)$  is

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

That is, if vector  $\mathbf{v}$  were positioned with its initial point at the origin, then its terminal point would be at  $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ .

### Example

The vector  $\mathbf{v}$  with initial point  $(-1, 4, 5)$  and final point  $(4, -3, 2)$  is

$$\mathbf{v} = (4 - (-1), -3 - 4, 2 - 5) = (5, -7, -3)$$

### Parametric Equations for a Line in 3-space

The line through the point  $(x_0, y_0, z_0)$  and parallel to the non-zero vector  $\mathbf{v} = (a, b, c)$  has parametric equations

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

### Example

The line through  $(2, -1, 3)$  and parallel to the vector  $\mathbf{v} = (3, -7, 4)$  has parametric equations



$$x = 2+3t$$

$$y = -1-7t$$

$$z = 3+4t$$

Notice that when  $t = 0$ , we are at the point  $(2, -1, 3)$ . As  $t$  increases or decreases from 0, we move away from this point parallel to the direction indicated by  $(3, -7, 4)$ .

If you know two points  $p_1 = (x_1, y_1, z_1)$  and  $p_2 = (x_2, y_2, z_2)$  that a line passes through, you can find a parameterization for the line. First, find the displacement vector  $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ . then write down parametric equations for the line through either  $p_1$  or  $p_2$  and parallel to  $\mathbf{v}$ .

### Equation of a Plane in 3-space

The equation of the plane containing the point  $(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = (a, b, c)$  is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Thus, the graph of the equation

$$ax + by + cz = d$$

is a plane with normal vector  $(a, b, c)$ .

### Example

The equation of the plane containing  $(2, 4, -1)$  and normal to the vector  $\mathbf{n} = (3, 5, -2)$  is

$$3(x-2) + 5(y-4) - 2(z-(-1)) = 0$$

Simplifying,

$$3x + 5y - 2z = 28.$$

With a little extra work, we can use this procedure to find the equation of the plane defined by any three points. First, compute displacement vectors  $\mathbf{u}$  and  $\mathbf{v}$  between two pairs of these points. Then  $\mathbf{n} = \mathbf{u} \times \mathbf{v}$  is normal to the plane. Now, use one of the points and the vector  $\mathbf{n} = \mathbf{u} \times \mathbf{v}$  to obtain the equation of the plane.

### Key Concepts

- **Displacement Vector**

The displacement vector  $\mathbf{v}$  with initial point  $(x_1, y_1, z_1)$  and terminal point  $(x_2, y_2, z_2)$  is  $\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ .

- **Parametric Equations for a line in 3-space**

The line through the point  $(x_0, y_0, z_0)$  and parallel to the non-zero vector  $\mathbf{v} = (a, b, c)$  has parametric equations

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

- **Equation of a plane in 3-space**

The equation of the plane containing the point  $(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = (a, b, c)$  is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

## Multiple Integration

Recall our definition of the definite integral of a function of *single* variable:

Let  $f(x)$  be defined on  $[a, b]$  and let  $x_0, x_1, \dots, x_n$  be a partition of  $[a, b]$ . For each  $[x_{i-1}, x_i]$ , let  $x_i^* \in [x_{i-1}, x_i]$ . Then

$$\int_a^b f(x) \, dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

We can extend this definition to define the integral of a function of two or more variables.

### Double Integral of a Function of Two

#### Variables

Let  $f(x, y)$  be defined on a closed and bounded region  $R$  of the  $xy$ -plane. Set up a grid of vertical and horizontal lines in the  $xy$ -plane to form an **inner partition** of  $R$  into  $n$  rectangular subregions  $R_k$  of area  $\Delta A_k$ , each of which lies entirely in  $R$ . (Ignore the rectangles that are not entirely contained in  $R$ ) Choose a point  $(x_k^*, y_k^*)$  in each subregion  $R_k$ . The sum

$$\sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$$

is called a **Riemann Sum**. In the limit as we make our grid more and more dense, we define the **double integral of  $f(x,y)$  over  $R$**  as

$$\iint_R f(x,y) dA = \lim_{\max \Delta A_k \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta A_k$$

### Notes

- If this limit exists, we say that  $f$  is **integrable** over the **region of integration  $R$** .
- If  $f$  is continuous on  $R$ , then  $f$  is integrable over  $R$ .